Primordial perturbations in tachyonic power-law inflation

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In this work we determine the power spectrum of the gravitational potential of the primordial fluctuations for an inflationary model whose *inflaton* is a non-canonical scalar field of the tachyon-type. The respective background field equations for an inverse-square potential produce a power-law inflation, and it is explicitly shown that for such a potential the power spectrum tends to be scale-independent for highly accelerated regimes in the inflationary expansion.

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About the model.— The tachyon field has received some attention in cosmology, being applied to produce an accelerated expansion of the universe. It can play an important role in inflationary models [1–8] as well as in the present accelerated expansion, simulating the effect of the dark energy [3, 9–13]. The inverse-square potential we consider in our model was firstly proposed in [1] and it can promote a power-law accelerated expansion. The analysis of the dynamics of the respective potential was performed in references [3, 9–12]. As follows, we will analyze the primordial perturbations of such a model through the explicit calculation of the power spectrum of the gravitational potential.

Background field equations.— The model we consider here is described by the following action

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{2} + V\sqrt{1 - g^{\mu\nu}\partial_\nu\varphi\partial_\nu\varphi}\right\}, \qquad (1)$$

where R is the Ricci scalar and φ is a non-canonical scalar field of the tachyon-type with the self-interaction potential $V(\varphi)$. For a flat Friedmann-Robertson-Walker (F-R-W) metric, the background energy density and pressure of the tachyon condensate are

$$\rho_0 = \frac{V}{\sqrt{1-\dot{\varphi}_0^2}}, \qquad p_0 = -V\sqrt{1-\dot{\varphi}_0^2}, \qquad (2)$$

respectively. The corresponding Friedmann equation and time evolution equation of the field read

$$H^2 = \frac{V}{3\sqrt{1-\dot{\varphi}_0^2}}, \qquad \frac{\ddot{\varphi}_0}{1-\dot{\varphi}_0^2} + 3H\dot{\varphi}_0 + \frac{1}{V}\frac{dV}{d\varphi_0} = 0, \quad (3)$$

respectively, where $H = \dot{a}/a$ is the Hubble parameter, a is the scale factor and the dot denotes derivative with respect to time.

By considering an inverse-square potential

$$V = \frac{\lambda}{\varphi_0^2}, \qquad (4)$$

with λ being a constant, the above system is satisfied by the solution

$$a(t) = a_0(t+c_1)^n, \qquad \varphi_0(t) = \sqrt{\frac{2}{3n}}(t+c_1). \qquad (5)$$

Here a_0, c_1 and n are constants and the following relation between n and λ holds

$$n = \frac{1}{6}\left(2+\sqrt{4+9\lambda^2}\right). \qquad (6)$$

Such a model describes a power-law inflation if $\lambda > 2/\sqrt{3}$. The speed of sound corresponding to this solution reads

$$c_s = \sqrt{\frac{3n-2}{3n}}. \qquad (7)$$

Cosmological perturbations.— Since the fluid of the model does not present anisotropic stress, the small scalar perturbations in the flat F-R-W line element in the longitudinal gauge [14–16] are represented by

$$ds^2 = a^2(\eta)\left[(1+2\Phi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right]. \qquad (8)$$

Above η is the conformal time, defined as $d\eta = dt/a$, and Φ is the gravitational potential of the perturbation.

Introducing the Mukhanov-Sasaki variables [15, 16]

$$u = \frac{2\Phi}{\sqrt{\rho_0+p_0}}, \qquad \theta = \frac{1}{\sqrt{3}}\left(\frac{1}{a}\right)\left(1+\frac{p_0}{\rho_0}\right)^{-\frac{1}{2}}, \qquad (9)$$

the differential equation that describes the perturbations reads

$$u'' - c_s^2\nabla^2 u - \frac{\theta''}{\theta}u = 0, \qquad (10)$$

where the prime denotes derivative with respect to the conformal time.

Rewriting (10) such that u is a function of the scale factor, we can set it in the form

$$a^2\frac{d^2u}{da^2} + a\left(2+\frac{\dot{H}}{H^2}\right)\frac{du}{da} - \frac{c_s^2}{a^2H^2}\nabla^2 u + \frac{\dot{H}}{H^2}u = 0. \quad (11)$$

If we apply the background solution to the above equation and consider a plane-wave solution in the form $u_{\mathbf{k}}e^{-i\mathbf{k}.\mathbf{x}}$, one has

$$a^2\frac{d^2u_{\mathbf{k}}}{da^2} + (2p+1)\,a\frac{du_{\mathbf{k}}}{da} + \left[k^2\alpha^2 a^{-4p} - \frac{1}{n}\right]u_{\mathbf{k}} = 0, \quad (12)$$

where

$$p = \frac{n-1}{2n}, \qquad \alpha^2 = \frac{c_s^2}{a_0^{2/n} n^2}. \tag{13}$$

Equation (12) is one of the forms of the Bessel equation, whose solution can be written as

$$u_{\mathbf{k}} = a^{-p}\left[A_{\mathbf{k}} J_q\left(\beta k a^{-2p}\right) + B_{\mathbf{k}} Y_q\left(\beta k a^{-2p}\right)\right], \tag{14}$$

which is valid for integer q. For non integer q, its form is

$$u_{\mathbf{k}} = a^{-p}\left[A_{\mathbf{k}} J_q\left(\beta k a^{-2p}\right) + B_{\mathbf{k}} J_{-q}\left(\beta k a^{-2p}\right)\right]. \tag{15}$$

Above $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are constants and

$$q = \frac{n+1}{2(n-1)}, \qquad \beta = \frac{n\alpha}{(n-1)}. \tag{16}$$

Firstly, let us calculate the asymptotic forms of the solution for integer q. For short-wavelength perturbations, which satisfy $ka^{-2p} \gg 1$, one obtains by using the asymptotic forms of the Bessel functions for a large argument in (14)

$$u_{\mathbf{k}} \simeq C_{\mathbf{k}} e^{i\beta k (a^{-2p} - a_i^{-2p})}. \tag{17}$$

We have set $B_{\mathbf{k}} = iA_{\mathbf{k}}$ and defined

$$C_{\mathbf{k}} = \sqrt{\frac{2}{\pi\beta k}} A_{\mathbf{k}} e^{-i\xi} e^{i\beta k a_i^{-2p}}, \qquad \xi = \frac{\pi n}{2(n-1)}, \tag{18}$$

being $a_i = a(t_i)$ or $a_i = a(\eta_i)$, where t_i and η_i stand for an initial instant. These redefinitions for the constants will facilitate the fixing of the initial conditions later. For long-wavelength perturbations, when $ka^{-2p} \ll 1$, one has by employing the asymptotic forms of the Bessel functions for a small argument in (14)

$$u_{\mathbf{k}} \simeq C_{1\mathbf{k}} a^{\frac{1}{n}} + C_{2\mathbf{k}} a^{-1}. \tag{19}$$

The constants were redefined as

$$C_{1\mathbf{k}} = -\frac{i}{\pi} A_{\mathbf{k}} \Gamma(q) \left(\frac{\beta k}{2}\right)^{-q}, \quad C_{2\mathbf{k}} = \frac{A_{\mathbf{k}}}{\Gamma(1+q)} \left(\frac{\beta k}{2}\right)^{q}. \tag{20}$$

Now, proceeding in the same way as we did above, from equation (15) one calculates the asymptotic forms of the solution for non integer q. For the limit of short-wavelength perturbations, we get by setting $B_{\mathbf{k}} = iA_{\mathbf{k}}$ for convenience

$$u_{\mathbf{k}} \simeq D_{\mathbf{k}}\left[\cos\left(\beta k a^{-2p} - \xi\right) + i\cos\left(\beta k a^{-2p} - \xi + \pi q\right)\right], \tag{21}$$

where

$$D_{\mathbf{k}} = \sqrt{\frac{2}{\pi\beta k}} A_{\mathbf{k}}. \tag{22}$$

For the limit of long-wavelength perturbations, we have

$$u_{\mathbf{k}} \simeq D_{1\mathbf{k}} a^{\frac{1}{n}} + D_{2\mathbf{k}} a^{-1}, \tag{23}$$

with

$$D_{1\mathbf{k}} = -\frac{iA_{\mathbf{k}}}{\Gamma(1-q)} \left(\frac{\beta k}{2}\right)^{-q}, \quad D_{2\mathbf{k}} = \frac{A_{\mathbf{k}}}{\Gamma(1+q)} \left(\frac{\beta k}{2}\right)^{q}. \tag{24}$$

Power spectrum for integer q.— According to the current inflationary theory, the primordial quantum fluctuations generated the seeds for the large scale structures. Thus the primordial power spectrum must be consistent with the minimal fluctuations of energy allowed by quantum mechanics, i.e., we have to preserve the fluctuations of the vacuum. Such a requirement is satisfied if the initial conditions for the variable $u_{\mathbf{k}}$ present the forms [15, 16]

$$u_{\mathbf{k}i} = -\frac{i}{\sqrt{c_s}} k^{-\frac{3}{2}}, \qquad u'_{\mathbf{k}i} = \sqrt{c_s} k^{-\frac{1}{2}}. \tag{25}$$

To satisfy these initial conditions, from (17) we determine that $C_{\mathbf{k}}$ must have the following value

$$C_{\mathbf{k}} = -i\frac{a_0^{1/n}\sqrt{\lambda}}{2c_s k^{\frac{3}{2}}}. \tag{26}$$

So, the $u_{\mathbf{k}}$ for short wave-length perturbations that satisfy the minimum fluctuations of energy reads

$$u_{\mathbf{k}} \simeq -\frac{i}{\sqrt{c_s} k^{\frac{3}{2}}} e^{i\beta k (a^{-2p} - a_i^{-2p})}. \tag{27}$$

The corresponding power spectrum is

$$\delta_\Phi^2(k,a) \simeq \frac{\lambda}{16\pi^2 c_s^2} \left(\frac{a}{a_0}\right)^{-\frac{2}{n}}, \tag{28}$$

which is scale-independent.

From result (26), through the use of $(18)_1$ and (20), we determine $A_{\mathbf{k}}$ and the constants of solution (19). Since $k^2 \ll a^{-2p}$ for this solution, one can keeps only its first term, obtaining the respective $u_{\mathbf{k}}$

$$u_{\mathbf{k}} \simeq -\frac{\Gamma(q)}{\sqrt{\pi c_s} k^{1+q}} \left(\frac{\beta}{2}\right)^{\frac{1}{2}-q} a^{\frac{1}{n}}. \tag{29}$$

This furnishes the following power spectrum

$$\delta_\Phi^2(k,a) \simeq \frac{\lambda a_0^{2/n} \Gamma(q)^2}{16\pi^3 c_s^2} \left(\frac{\beta}{2}\right)^{1-2q} k^{1-2q}, \tag{30}$$

valid for long wave-length perturbations. Such a power spectrum is scale-dependent for all the q. Note that this result is valid only for integer q.

Power spectrum for non integer q.— For this case, by using (21), the initial condition (25) is satisfied if $D_{\mathbf{k}}$ has the form

$$D_{\mathbf{k}} = -\frac{i a_0^{1/n}\sqrt{\lambda}}{2c_s k^{\frac{3}{2}}} [\cos(\beta k a_i^r - \xi) + i\cos(\beta k a_i^r - \xi + \pi q)]^{-1}. \tag{31}$$

Thus the adequate $u_{\mathbf{k}}$ for the limit of short wave-length perturbations is

$$u_{\mathbf{k}} \simeq -\frac{i}{\sqrt{c_s}k^{\frac{3}{2}}}\left[\frac{\cos\left(\beta k a^r - \xi\right) + i\cos\left(\beta k a^r - \xi + \pi q\right)}{\cos\left(\beta k a_i^r - \xi\right) + i\cos\left(\beta k a_i^r - \xi + \pi q\right)}\right]. \tag{32}$$

The power spectrum for this limit is

$$\delta_\Phi^2(k,a) \simeq \frac{\lambda}{16\pi^2 c_s^2}\times \left[\frac{\cos^2\left(\beta k a^r - \xi + \pi q\right) + \cos^2\left(\beta k a^r - \xi\right)}{\cos^2\left(\beta k a_i^r - \xi + \pi q\right) + \cos^2\left(\beta k a_i^r - \xi\right)}\right]\left(\frac{a}{a_0}\right)^{-\frac{2}{n}}. \tag{33}$$

From result (31), and (22) through (24), the asymptotic form (23), by despising its second term, renders

$$u_{\mathbf{k}} \simeq -\frac{i\sqrt{2\pi}\,(\beta/2)^{\frac{1}{2}-q}\,k^{-1-q}}{2\sqrt{c_s}\,\Gamma(1-q)}\times \left[\cos\left(\beta k a_i^r - \xi\right) + i\cos\left(\beta k a_i^r - \xi + \pi q\right)\right]^{-1} a^{\frac{1}{n}}. \tag{34}$$

Hence the respective power spectrum, valid for long wave-length perturbations, reads

$$\delta_\Phi^2(k,a) \simeq \frac{\lambda a_0^{2/n}\,(\beta/2)^{1-2q}}{32\pi c_s^2\Gamma^2(1-q)}\times \left[\cos^2\left(\beta k a_i^r - \xi + \pi q\right) + \cos^2\left(\beta k a_i^r - \xi\right)\right]^{-1} k^{1-2q}. \tag{35}$$

Analysing the quantities (33) and (35), we can infer that they tend to be scale-independent when $q \to 1/2$. On the other hand, from expressions $(5)_1$ and $(16)_1$ one has the following relations connecting q, n and the scale factor

$$q = \frac{n+1}{2(n-1)}, \qquad a(t) = a_0(t+c_1)^n, \tag{36}$$

which show that the situation $q \to 1/2$ occurs when n is sufficiently large. Note that this holds when the rate of the inflationary expansion is highly accelerated. Thus the present inflationary model produces a power spectrum which is progressively more scale-independent whenever the primordial expansion of the universe is more accelerated.

[1] A. Feinstein, Phys. Rev. D **66**, 063511 (2002).
[2] M. Sami, P. Chingangbam, and T. Qureshi , Phys. Rev. D **66**, 043530 (2002).
[3] L. R. W. Abramo, and F. Finelli, Phys. Lett. B. **575**, 165 (2003).
[4] C. Campuzano, S. del Campo, and R. Herrera, Phys. Lett. B **633**, 149 (2006).
[5] V. H. Cárdenas, Phys. Rev. D **73**, 103512 (2006).
[6] S. Chattopadhyay, U. Debnath, and G. Chattopadhyay, Astroph. Space Science **314**, 41 (2008).
[7] S. del Campo, R. Herrera, and A. Toloza, Phys. Rev. D **79**, 083507 (2009).
[8] R. K. Jain, P. Chingangbam, and L. Sriramkumar, Nucl. Phys. B **852**, 366 (2011).
[9] T. Padmanabhan, Phys. Rev. D **66**, 021301(R) (2002).
[10] J. S. Bagla, H. K. Jassal, and T. Padmanabhan, Phys. Rev. D **67**, 063504 (2003).
[11] J.M. Aguirregabiria, and R. Lazkoz, Phys. Rev. D **69**, 123502 (2004).
[12] G. Calcagni, and A. R. Liddle, Phys. Rev. D **74**, 043528 (2006).
[13] P. P. Avelino, L. Losano, and J. J. Rodrigues, Phys. Lett. B **699**, 10 (2011).
[14] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[15] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, R. H. Phys. Rep. **215**, 203 (1992).
[16] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, 2005).